

STABILIZATION OF THE HOMOTOPY GROUPS OF THE MODULI SPACES OF k -HIGGS BUNDLES

February 24th, 2017

*Ronald A. Zúñiga-Rojas*¹

Centro de Investigaciones Matemáticas y Metamatemáticas CIMM
Universidad de Costa Rica UCR
San José 11501, Costa Rica
e-mail: ronald.zunigarojas@ucr.ac.cr

Abstract. The work of Hausel proves that the Białynicki-Birula stratification of the moduli space of rank two Higgs bundles coincides with its Shatz stratification. He uses that to estimate some homotopy groups of the moduli space of k -Higgs bundles of rank two. Unfortunately, those two stratifications do not coincide in general. Here, the objective is to present a different proof of the stabilization of the homotopy groups of $\mathcal{M}^k(2, d)$, and generalize it to $\mathcal{M}^k(3, d)$, the moduli spaces of k -Higgs bundles of degree d , and ranks two and three respectively, using the results from the works of Hausel and Thaddeus, among other tools.

Keywords: Moduli of Higgs Bundles, Variations of Hodge Structures, Vector Bundles.

MSC classes: Primary 14H60; Secondaries 14D07, 55Q52.

Introduction

In this work, we estimate some homotopy groups of the moduli space of rank three k -Higgs bundles $\mathcal{M}^k(3, d)$ on a Riemann surface X of genus $g \geq 2$. This space was first introduced by Hitchin [19]; and then, it was worked by Hausel [15], where he estimated some of the homotopy groups working the particular case of rank two. The co-prime condition $\text{GCD}(3, d) = 1$ implies that the moduli space $\mathcal{M}^k(3, d)$ is smooth. We shall do the estimate with Higgs bundles of fixed determinant $\det(E) = \Lambda \in \mathcal{N}$ to ensure that $\mathcal{N}(3, d)$ and $\mathcal{M}(3, d)$ are simply connected, and so, the group action $\pi_1(\mathcal{M}_\Lambda^k) \curvearrowright \pi_n(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k)$ will be trivial.

Hausel [15] estimates the homotopy groups $\pi_n(\mathcal{M}^k(2, 1))$ using two main tools: first the coincidence mentioned before between the Białynicki-Birula stratification and the Shatz stratification; and second, the well-behaved embeddings $\mathcal{M}^k(2, 1) \hookrightarrow \mathcal{M}^{k+1}(2, 1)$. These inclusions are also well-behaved in general for $\text{GCD}(r, d) = 1$; nevertheless, those two stratifications mentioned above do not coincide in general (see for instance [10]).

In this paper, our estimate is based on the embeddings $\mathcal{M}^k(3, d) \hookrightarrow \mathcal{M}^{k+1}(3, d)$ and their good behavior, notwithstanding the non-coincidence between stratifications. The paper is organized as follows: in section 1 we recall some facts about vector bundles and Higgs bundles; in section 2, we present the cohomology ring $H^n(\mathcal{M}^k)$; in section 3, we discuss the most relevant results about the cohomology and the homotopy of the moduli spaces \mathcal{M}^k ; finally, in section 4 we present and prove the main result.

¹Supported by Universidad de Costa Rica through CIMM (Centro de Investigaciones Matemáticas y Metamatemáticas), Project 820-B5-202. This work is based on the Ph.D. Project [30] called “Homotopy Groups of the Moduli Space of Higgs Bundles”, supported by FEDER through Programa Operacional Factores de Competitividade-COMPETE, and also supported by FCT (Fundação para a Ciência e a Tecnologia) through the projects PTDC/MAT-GEO/0675/2012 and PEst-C/MAT/UI0144/2013 with grant reference SFRH/BD/51174/2010.

1 Preliminary definitions

Let X be a closed Riemann surface of genus $g \geq 2$ and let $K = T^*X$ be the canonical line bundle of X . Note that, algebraically, X is also a nonsingular complex projective algebraic curve.

Definition 1.1. A *Higgs bundle* over X is a pair (E, Φ) where $E \rightarrow X$ is a holomorphic vector bundle and $\Phi: E \rightarrow E \otimes K$ is an endomorphism of E twisted by K , which is called a *Higgs field*. Note that $\Phi \in H^0(X; \text{End}(E) \otimes K)$.

Definition 1.2. For a vector bundle $E \rightarrow X$, we denote the *rank* of E by $\text{rk}(E) = r$ and the *degree* of E by $\deg(E) = d$. Then, for any smooth bundle $E \rightarrow X$ the *slope* is defined to be

$$\mu(E) := \frac{\deg(E)}{\text{rk}(E)} = \frac{d}{r}. \quad (1.1)$$

A vector bundle $E \rightarrow X$ is called *semistable* if $\mu(F) \leq \mu(E)$ for any F such that $0 \subsetneq F \subsetneq E$. Similarly, a vector bundle $E \rightarrow X$ is called *stable* if $\mu(F) < \mu(E)$ for any nonzero proper subbundle $0 \subsetneq F \subsetneq E$. Finally, E is called *polystable* if it is the direct sum of stable subbundles, all of the same slope.

Definition 1.3. A subbundle $F \subset E$ is said to be Φ -*invariant* if $\Phi(F) \subset F \otimes K$. A Higgs bundle is said to be *semistable* [respectively, *stable*] if $\mu(F) \leq \mu(E)$ [resp., $\mu(F) < \mu(E)$] for any nonzero Φ -invariant subbundle $F \subseteq E$ [resp., $F \subsetneq E$]. Finally, (E, Φ) is called *polystable* if it is the direct sum of stable Φ -invariant subbundles, all of the same slope.

Fixing the rank $\text{rk}(E) = r$ and the degree $\deg(E) = d$ of a Higgs bundle (E, Φ) , the isomorphism classes of polystable bundles are parametrized by a quasi-projective variety: the moduli space $\mathcal{M}(r, d)$. Constructions of this space can be found in the work of Hitchin [19], using gauge theory, or in the work of Nitsure [26], using algebraic geometry methods.

An important feature of $\mathcal{M}(r, d)$ is that it carries an action of \mathbb{C}^* : $z \cdot (E, \Phi) = (E, z \cdot \Phi)$. According to Hitchin [19], (\mathcal{M}, I, Ω) is a Kähler manifold, where I is its complex structure and Ω its corresponding Kähler form. Furthermore, \mathbb{C}^* acts on \mathcal{M} biholomorphically with respect to the complex structure I by the action mentioned above, where the Kähler form Ω is invariant under the induced action $e^{i\theta} \cdot (E, \Phi) = (E, e^{i\theta} \cdot \Phi)$ of the circle $\mathbb{S}^1 \subset \mathbb{C}^*$. Besides, this circle action is Hamiltonian, with proper momentum map $f: \mathcal{M} \rightarrow \mathbb{R}$ defined by:

$$f(E, \Phi) = \frac{1}{2\pi} \|\Phi\|_{L^2}^2 = \frac{i}{2\pi} \int_X \text{tr}(\Phi \Phi^*), \quad (1.2)$$

where Φ^* is the adjoint of Φ with respect to a hermitian metric on E , and f has finitely many critical values.

There is another important fact mentioned by Hitchin (see the original version in Frankel [7], and its application to Higgs bundles in Hitchin [19]): the critical points of f are exactly the fixed points of the circle action on \mathcal{M} .

If $(E, \Phi) = (E, e^{i\theta} \Phi)$ then $\Phi = 0$ with critical value $c_0 = 0$. The corresponding critical submanifold is $F_0 = f^{-1}(c_0) = f^{-1}(0) = \mathcal{N}$, the moduli space of stable bundles. On the other hand, when $\Phi \neq 0$, there is a type of algebraic structure for Higgs bundles introduced by Simpson [28]: a *variation of Hodge structure*, or simply a *VHS*, for a Higgs bundle (E, Φ) is a decomposition:

$$E = \bigoplus_{j=1}^n E_j \quad \text{such that} \quad \Phi: E_j \rightarrow E_{j+1} \otimes K \text{ for } 1 \leq j \leq n-1. \quad (1.3)$$

It has been proved by Simpson [29] that the fixed points of the circle action on $\mathcal{M}(r, d)$, and so, the critical points of f , are these Variations of the Hodge Structure, VHS, where the critical values $c_\lambda = f(E, \Phi)$ will depend on the degrees d_j of the components $E_j \subset E$. By Morse theory, we can stratify \mathcal{M} in such a way that there is a nonzero critical submanifold $F_\lambda := f^{-1}(c_\lambda)$ for each nonzero critical value $0 \neq c_\lambda = f(E, \Phi)$ where (E, Φ) represents a fixed point of the circle action, or equivalently, a VHS. We then say that (E, Φ) is an (r_1, \dots, r_n) -VHS, where $r_j = \text{rk}(E_j) \forall j$.

The reader may consult the works of Bradlow and García-Prada [4]; Bradlow, García-Prada and Gothen [5]; and Muñoz, Ortega and Vázquez-Gallo [25] for the details on the results summarized below.

Definition 1.4. Holomorphic Triples:

- i. A *holomorphic triple* on X is a triple $T = (E_1, E_2, \phi)$ consisting of two holomorphic vector bundles $E_1 \rightarrow X$ and $E_2 \rightarrow X$ and a homomorphism $\phi: E_2 \rightarrow E_1$, i.e., an element $\phi \in H^0(\text{Hom}(E_2, E_1))$.
- ii. A *homomorphism* from a triple $T' = (E'_1, E'_2, \phi')$ to another triple $T = (E_1, E_2, \phi)$ is a commutative diagram of the form:

$$\begin{array}{ccc} E'_1 & \xrightarrow{\phi'} & E'_2 \\ \downarrow & & \downarrow \\ E_1 & \xrightarrow{\phi} & E_2 \end{array}$$

where the vertical arrows represent holomorphic maps.

- iii. $T' \subset T$ is a *subtriple* if the sheaf homomorphisms $E'_1 \rightarrow E_1$ and $E'_2 \rightarrow E_2$ are injective. As usual, a subtriple is called *proper* if $0 \neq T' \subsetneq T$.

Definition 1.5. σ -Stability, σ -Semistability and σ -Polystability:

- i. For any $\sigma \in \mathbb{R}$, the σ -degree and the σ -slope of $T = (E_1, E_2, \phi)$ are defined as:

$$\deg_\sigma(T) := \deg(E_1) + \deg(E_2) + \sigma \cdot \text{rk}(E_2),$$

and

$$\begin{aligned} \mu_\sigma(T) &:= \frac{\deg_\sigma(T)}{\text{rk}(E_1) + \text{rk}(E_2)} = \frac{\deg(E_1) + \deg(E_2) + \sigma \cdot \text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)} \\ &= \mu(E_1 \oplus E_2) + \sigma \frac{\text{rk}(E_2)}{\text{rk}(E_1) + \text{rk}(E_2)}. \end{aligned}$$

- ii. T is then called σ -stable [resp., σ -semistable] if $\mu_\sigma(T') < \mu_\sigma(T)$ [resp., $\mu_\sigma(T') \leq \mu_\sigma(T)$] for any proper subtriple $0 \neq T' \subsetneq T$.
- iii. A triple is called σ -polystable if it is the direct sum of σ -stable triples of the same σ -slope.

Now we may use the following notation for moduli spaces of triples:

- i. Denote $\mathbf{r} = (r_1, r_2)$ and $\mathbf{d} = (d_1, d_2)$, and then regard

$$\mathcal{N}_\sigma = \mathcal{N}_\sigma(\mathbf{r}, \mathbf{d}) = \mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$$

as the moduli space of σ -polystable triples $T = (E_1, E_2, \phi)$ such that $\text{rk}(E_j) = r_j$ and $\deg(E_j) = d_j$.

- ii. Denote by $\mathcal{N}_\sigma^s = \mathcal{N}_\sigma^s(\mathbf{r}, \mathbf{d})$ the subspace of σ -stable triples.

- iii. Call $(\mathbf{r}, \mathbf{d}) = (r_1, r_2, d_1, d_2)$ the type of the triple $T = (E_1, E_2, \phi)$.

We mention the moduli space $\mathcal{N}_\sigma(r_1, r_2, d_1, d_2)$ of σ -stable triples because they are close related to some of the critical submanifolds F_λ .

Definition 1.6. Fix a point $p \in X$, and let $L_p = \mathcal{O}_X(p)$ be the associated line bundle to the divisor $p \in \text{Sym}^1(X) = X$. A k -Higgs bundle (or Higgs bundle with poles of order k) is a pair (E, Φ^k) where:

$$E \xrightarrow{\Phi^k} E \otimes K \otimes L_p^{\otimes k}$$

and where the morphism $\Phi^k \in H^0(X, \text{End}(E) \otimes K \otimes L_p^{\otimes k})$ is what we call a *Higgs field with poles of order k* . The moduli space of k -Higgs bundles of rank r and degree d is denoted by $\mathcal{M}^k(r, d)$. For simplicity, we will suppose that $\text{GCD}(r, d) = 1$, and so, $\mathcal{M}^k(r, d)$ will be smooth.

There is an embedding

$$i_k : \mathcal{M}^k(r, d) \rightarrow \mathcal{M}^{k+1}(r, d) : [(E, \Phi^k)] \mapsto [(E, \Phi^k \otimes s_p)]$$

where $0 \neq s_p \in H^0(X, L_p)$ is a nonzero fixed section of L_p .

All the results mentioned for $\mathcal{M}(r, d)$, holds also for $\mathcal{M}^k(r, d)$.

2 Generators for the Cohomology Ring

According to Hausel and Thaddeus [16, (4.4)], there is a universal family (\mathbb{E}^k, Φ^k) over $X \times \mathcal{M}^k$ where

$$\begin{cases} \mathbb{E}^k & \rightarrow X \times \mathcal{M}^k(r, d) \\ \Phi^k & \in H^0(\text{End}(\mathbb{E}^k) \otimes \pi_2^*(K \otimes L_p^{\otimes k})) \end{cases}$$

and from now on, we will refer (\mathbb{E}^k, Φ^k) as a *universal k -Higgs bundle*. Note that (\mathbb{E}^k, Φ^k) satisfies the *Universal Property*: in general, for any family (\mathbb{F}^k, Ψ^k) over $X \times T$, there is a morphism $\eta : T \rightarrow \mathcal{M}^k$ such that $\eta^*(\mathbb{E}^k, \Phi^k) = (\mathbb{F}^k, \Psi^k)$. It means that, for $T = \mathcal{M}^k$ whenever exists (\mathbb{F}^k, Ψ^k) such that

$$(\mathbb{E}^k, \Phi^k)_P \cong (\mathbb{F}^k, \Psi^k)_P \quad \forall P = (E, \Phi^k) \in \mathcal{M}^k(r, d),$$

then, there exists a unique bundle morphism $\xi : \mathbb{F}^k \rightarrow \mathbb{E}^k$ such that

$$\begin{array}{ccc} \mathbb{F}^k & \xrightarrow{\quad \exists! \xi \quad} & \mathbb{E}^k \\ p_2 \searrow & & \swarrow p_1 \\ & X \times \mathcal{M}^k(r, d) & \end{array} \quad (2.1)$$

commutes: $p_2 = p_1 \circ \xi$.

The universal bundle extends then to the following: if (\mathbb{E}^k, Φ^k) and (\mathbb{F}^k, Ψ^k) are families of stable k -Higgs bundles parametrized by $\mathcal{M}^k(r, d)$, such that $(\mathbb{E}^k, \Phi^k)_P \cong (\mathbb{F}^k, \Psi^k)_P$ for all $P = (E, \Phi^k) \in \mathcal{M}^k(r, d)$, then there is a line bundle $\mathcal{L} \rightarrow \mathcal{M}^k(r, d)$ such that

$$(\mathbb{E}^k, \Phi^k) \cong (\mathbb{F}^k \otimes \pi_2^*(\mathcal{L}), \Psi^k \otimes \pi_2),$$

where $\pi_2: X \times \mathcal{M}^k(r, d) \rightarrow \mathcal{M}^k(r, d)$ is the natural projection and $\Phi^k \cong \Psi^k \otimes \pi_2(\sigma_P)$. For more details, see Hausel and Thaddeus [16, (4.2)].

If we consider the embedding $i_k: \mathcal{M}^k(r, d) \rightarrow \mathcal{M}^{k+1}(r, d)$ for general rank, we get that:

Proposition 2.1. *Let (\mathbb{E}^k, Φ^k) be a universal Higgs bundle. Then:*

$$(\text{Id}_X \times i_k)^*(\mathbb{E}^{k+1}) \cong \mathbb{E}^k.$$

Proof. Note that

$$(\mathbb{E}^k, \Phi^k \otimes \pi_1^*(s_p)) \rightarrow X \times \mathcal{M}^k$$

is a family of $(k+1)$ -Higgs bundles on X , where $\pi_1: X \times \mathcal{M}^k \rightarrow X$ is the natural projection. So, by the universal property:

$$(\mathbb{E}^k, \Phi^k \otimes \pi_1^*(s_p)) = j^*(\mathbb{E}^{k+1}, \Phi^{k+1})$$

where

$$\begin{aligned} j: X \times \mathcal{M}^k &\rightarrow X \times \mathcal{M}^{k+1} \\ (x, (E, \Phi^k)) &\mapsto (x, (E, \Phi^k \otimes s_p)). \end{aligned}$$

□

We now describe a result of Markman [24]. Choose a basis:

$$\{x_1, \dots, x_{2g}, x_{2g+1}, x_{2g+2}\} \subset K(X) = K^0(X) \oplus K^1(X),$$

where $\{x_1, \dots, x_{2g}\} \subset K^1(X)$, and $\{x_{2g+1}, x_{2g+2}\} \subset K^0(X)$ and so, since there is a universal bundle $\mathbb{E}^k \rightarrow X \times \mathcal{M}^k$, we can get the Künneth decomposition (see Atiyah [1, Corollary 2.7.15]):

$$[\mathbb{E}^k] = \sum_{j=0}^{2g} x_j \otimes e_j^k$$

for $e_j^k \in K(\mathcal{M}^k)$, since $K(X \times \mathcal{M}^k) \cong K(X) \otimes K(\mathcal{M})$.

Then, Markman [24] considers the Chern classes $c_j(e_i^k) \in H^{2j}(\mathcal{M}^k, \mathbb{Z})$ for $e_i^k \in K(\mathcal{M}^k)$ and proves that:

Theorem 2.2 (Markman [24, Theorem 3]). *The cohomology ring $H^*(\mathcal{M}^k(r, d), \mathbb{Z})$ is generated by the Chern classes of the Künneth factors of the universal vector bundle.*

3 Previous Results

Let $i_k: \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding given by the map $(E, \Phi^k) \mapsto (E, \Phi^k \otimes s_p)$, where s_p is a fixed nonzero section of L_p . We want to prove that the map

$$\pi_j(i_k) : \pi_j(\mathcal{M}^k(r, d)) \rightarrow \pi_j(\mathcal{M}^{k+1}(r, d))$$

stabilizes as $k \rightarrow \infty$. But first, we need to present some previous results to conclude that.

Proposition 3.1. *Let $i_k: \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding mentioned above. Consider the K -classes $e_i^k \in K(\mathcal{M}^k)$. Then $i_k^*(c_j(e_i^{k+1})) = c_j(e_i^k)$.*

Proof. By 2.1, and by the naturality of the Chern classes:

$$\sum_{j=0}^{2g} x_j \otimes e_j^k = [\mathbb{E}^k] = [(\text{Id}_X \times i_k)^*(\mathbb{E}^{k+1})] = \sum_{j=0}^{2g} x_j \otimes i_k^*(e_j^{k+1})$$

we have that $i_k^*(e_i^{k+1}) = e_i^k$ and hence $i_k^*(c_j(e_i^{k+1})) = c_j(e_i^k)$. \square

An immediate consequence will be

Corollary 3.2. *Let $i_k: \mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ be the embedding mentioned above. Then, the induced cohomology homomorphism $i_k^*: H^*(\mathcal{M}^{k+1}, \mathbb{Z}) \rightarrow H^*(\mathcal{M}^k, \mathbb{Z})$ is surjective.* \square

Definition 3.3. A *gauge transformation* is an automorphism of E . Locally, a gauge transformation $g \in \text{Aut}(E)$ is a $C^\infty(E)$ -function with values in $GL_r(\mathbb{C})$. A gauge transformation g is called *unitary* if g preserves a hermitian inner product on E . We will denote \mathcal{G} as the group of unitary gauge transformations. Atiyah and Bott [2] denote $\bar{\mathcal{G}}$ as the quotient of \mathcal{G} by its constant central $U(1)$ -subgroup. We will follow this notation too. Moreover, denote $B\mathcal{G}$ and $B\bar{\mathcal{G}}$ as the classifying spaces of \mathcal{G} and $\bar{\mathcal{G}}$, respectively.

There are a couple of results of Atiyah and Bott [2] that will be very useful for us:

Theorem 3.4 (Atiyah and Bott [2, (2.7.)]). *$H^*(B\mathcal{G}, \mathbb{Z})$ is torsion free and has Poincaré polynomial:*

$$P_t(B\mathcal{G}) = \frac{\left((1+t)(1+t^3)\right)^{2g}}{(1-t^2)^2(1-t^4)}.$$

Corollary 3.5 (Atiyah and Bott [2, (9.7.)]). *$H^*(B\bar{\mathcal{G}}, \mathbb{Z})$ is also torsion free with Poincaré polynomial:*

$$P_t(B\bar{\mathcal{G}}) = (1-t^2)P_t(B\mathcal{G}) = \frac{\left((1+t)(1+t^3)\right)^{2g}}{(1-t^2)(1-t^4)}.$$

Let $\mathcal{M}^\infty := \lim_{k \rightarrow \infty} \mathcal{M}^k = \bigcup_{k=0}^{\infty} \mathcal{M}^k$ be the direct limit of the spaces $\{\mathcal{M}^k(r, d)\}_{k=0}^{\infty}$.

Hausel and Thaddeus [16] prove that:

Theorem 3.6 (Hausel and Thaddeus [16, (9.7.)]). *The classifying space of $\bar{\mathcal{G}}$ is isomorphic to the direct limit of the spaces $\mathcal{M}^k(r, d)$:*

$$B\bar{\mathcal{G}} \cong \mathcal{M}^\infty = \lim_{k \rightarrow \infty} \mathcal{M}^k.$$

Proof. By the last corollary, $H^j(B\bar{\mathcal{G}}, \mathbb{Z}) \cong H^j(\mathcal{M}^\infty, \mathbb{Z}) \twoheadrightarrow H^j(\mathcal{M}^k, \mathbb{Z})$ must be surjective, since all the groups $H^j(B\bar{\mathcal{G}}, \mathbb{Z})$ are finitely generated free abelian groups. The result follows then from Corollary 3.2. \square

Theorem 3.7. $H^n(\mathcal{M}^k(2, d))$ and $H^n(\mathcal{M}^k(3, d))$ are torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$.

Proof. The proof uses the following result of Frankel [7, Corollary 1]:

$$F_\lambda^k \text{ is torsion free } \quad \forall \lambda \Leftrightarrow \mathcal{M}^k \text{ is torsion free.}$$

1. When $\text{rk}(E) = 2$, Hitchin notes that the nontrivial critical submanifolds, or $(1, 1)$ -VHS, are of the form

$$F_{d_1}^k = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \left| \begin{array}{l} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\ \text{rk}(E_1) = 1, \quad \text{rk}(E_2) = 1, \\ \varphi_{21}^k : E_1 \rightarrow E_2 \otimes K \otimes L_p^{\otimes k} \end{array} \right. \right\}$$

and $F_{d_1}^k$ is isomorphic to the moduli space of σ_H -stable triples $\mathcal{N}_{\sigma_H}(1, 1, \bar{d}, d_1)$, where σ_H is giving by $\sigma_H = \deg(K \otimes L_p^{\otimes k}) = 2g - 2 + k$ and $\bar{d} = d_2 + 2g - 2 + k - d_1$, by the map:

$$(E_1 \otimes E_2, \Phi^k) \mapsto (E_2 \otimes K \otimes L_p^{\otimes k}, E_1, \varphi_{21}^k).$$

Furthermore, by Hitchin [19], $\mathcal{N}_{\sigma_H}(1, 1, \bar{d}, d_1)$ is isomorphic to the cartesian product $\mathcal{J}^{d_1}(X) \times \text{Sym}^{\bar{d}-d_1}(X)$. Hence:

$$F_{d_1}^k \cong \mathcal{J}^{d_1}(X) \times \text{Sym}^{\bar{d}-d_1}(X)$$

which, by Macdonald [22, (12.3)], is indeed torsion free.

2. When $\text{rk}(E) = 3$, there are three kinds of nontrivial critical submanifolds:

2.1. $(1, 2)$ -VHS of the form

$$F_{d_1}^k = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \left| \begin{array}{l} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\ \text{rk}(E_1) = 1, \quad \text{rk}(E_2) = 2, \\ \varphi_{21}^k : E_1 \rightarrow E_2 \otimes K \otimes L_p^{\otimes k} \end{array} \right. \right\}.$$

In this case, there are isomorphisms between the $(1, 2)$ -VHS and the moduli spaces of triples $F_{d_1}^k \cong \mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$, where $\tilde{d}_1 = d_2 + 2(2g - 2 + k)$ and $\tilde{d}_2 = d_1$, and where the isomorphism is giving by a map similar to the mentioned above. By Muñoz, Ortega, Vázquez-Gallo [25], the flip loci $S_{\sigma_c}^+$ and $S_{\sigma_c}^-$ are free of torsion for all $\sigma_c \in I$, and so is $\mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$. Hence, $F_{d_1}^k$ is torsion free.

The fact that $\mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2)$ is torsion free since the flip loci are, follows from the next lemma:

Lemma 3.8. *Let M be a complex manifold, and let $\Sigma \subset M$ be a complex submanifold. Let \tilde{M} be the blow-up of M along Σ . Let $E = \mathbb{P}(N_{\Sigma/M})$ be the projectivized normal bundle of Σ in M , sometimes called exceptional divisor. Then*

$$H^*(\tilde{M}, \mathbb{Z}) \cong H^*(M, \mathbb{Z}) \oplus H^{*+2}(\Sigma, \mathbb{Z}) \oplus \dots \oplus H^{*+2n-2}(\Sigma, \mathbb{Z})$$

where n is the rank of $N_{\Sigma/M}$.

Proof. (Lemma 3.8)

It follows from the fact that the additive cohomology of the blow-up $H^*(\tilde{M}, \mathbb{Z})$, can be expressed as:

$$H^*(\tilde{M}) \cong \pi^* H^*(M) \oplus H^*(E)/\pi^* H^*(\Sigma)$$

(see for instance Griffiths and Harris [11, Chapter 4., Section 6.]), and the fact that $H^*(E)$ is a free module over $H^*(\Sigma)$ via the injective map $\pi^*: H^*(\Sigma) \rightarrow H^*(E)$ with basis

$$1, c, \dots, c^{n-1},$$

where $c \in H^2(E)$ is the first Chern class of the tautological line bundle along the fibres of the projective bundle $E \rightarrow \Sigma$ (see the general version at Husemoller [20, Chapter 17., Theorem 2.5.]). \square

2.2. (2, 1)-VHS of the form

$$F_{d_2}^k = \left\{ (E, \Phi^k) = (E_2 \oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \left| \begin{array}{l} \deg(E_2) = d_2, \quad \deg(E_1) = d_1, \\ \text{rk}(E_2) = 2, \quad \text{rk}(E_1) = 1, \\ \varphi_{21}^k : E_2 \rightarrow E_1 \otimes K \otimes L_p^{\otimes k} \end{array} \right. \right\}.$$

By symmetry, similar results can be obtained using the isomorphisms between the (2, 1)-VHS and the moduli spaces of triples: $F_{d_2}^k \cong \mathcal{N}_{\sigma_H(k)}(1, 2, \tilde{d}_1, \tilde{d}_2)$, and the dual isomorphisms

$$\mathcal{N}_{\sigma_H(k)}(2, 1, \tilde{d}_1, \tilde{d}_2) \cong \mathcal{N}_{\sigma_H(k)}(1, 2, -\tilde{d}_2, -\tilde{d}_1)$$

between moduli spaces of triples.

2.3. (1, 1, 1)-VHS of the form

$$F_{d_1 d_2 d_3}^k = \left\{ (E, \Phi^k) = (E_1 \oplus E_2 \oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21}^k & 0 & 0 \\ 0 & \varphi_{32}^k & 0 \end{pmatrix}) \left| \begin{array}{l} \deg(E_j) = d_j, \\ \text{rk}(E_j) = 1, \\ \varphi_{ij} : E_j \rightarrow E_i \otimes K \end{array} \right. \right\}.$$

Finally, we know that

$$F_{d_1 d_2 d_3}^k \xrightarrow{\cong} \text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X) \times \mathcal{J}^{d_3}(X)$$

$$(E, \Phi^k) \mapsto (\text{div}(\varphi_{21}^k), \text{div}(\varphi_{32}^k), E_3),$$

where $m_i = d_{i+1} - d_i + \sigma_H$, and so, by Macdonald [22, (12.3)] there is nothing to worry about torsion. \square

Using all the facts above, and denoting simply $\mathcal{M}^k = \mathcal{M}^k(r, d)$, Hausel and Thaddeus [16] conclude that:

Corollary 3.9 (Hausel and Thaddeus [16, (10.1)]). *If $H^*(\mathcal{M}^k, \mathbb{Z})$ is torsion free, then $H^*(B\bar{\mathcal{G}}, \mathbb{Z}) = \varprojlim H^*(\mathcal{M}^k, \mathbb{Z})$.*

And so, we may conclude also that:

Theorem 3.10. *For rank $r = 2$ and rank $r = 3$, $\forall j \in \mathbb{N}$, $\exists k_0 = k_0(j)$ such that*

$$i_k^* : H^j(\mathcal{M}^{k+1}, \mathbb{Z}) \xrightarrow{\cong} H^j(\mathcal{M}^k, \mathbb{Z}) \quad \forall k \geq k_0.$$

By the Universal Coefficient Theorem for Cohomology (see for instance Hatcher [13, Theorem 3.2. and Corollary 3.3.]), we would get

Lemma 3.11. *For rank $r = 2$ and rank $r = 3$, $\forall n \in \mathbb{N}$, $\exists k_0 = k_0(n)$ such that $H_j(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0 \ \forall k \geq k_0$ and $\forall j \leq n$.*

Proof. The embedding $i_k: \mathcal{M}^k(r, d) \rightarrow \mathcal{M}^{k+1}(r, d)$ is injective, and by Theorem 3.10, we know that $i_k^*: H^j(\mathcal{M}^k, \mathbb{Z}) \leftarrow H^j(\mathcal{M}^{k+1}, \mathbb{Z})$ is surjective $\forall k$. Hence, by the Universal Coefficient Theorem, we get that the following diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & \text{Ext}(H_{j-1}(\mathcal{M}^k), \mathbb{Z}) & \longrightarrow & H^j(\mathcal{M}^k, \mathbb{Z}) & \longrightarrow & \text{Hom}(H_j(\mathcal{M}^k), \mathbb{Z}) \longrightarrow 0 \\
 & & \uparrow (i_{k*})^* & & \uparrow i_k^* & & \uparrow (i_{k*})^* \\
 0 & \longrightarrow & \text{Ext}(H_{j-1}(\mathcal{M}^{k+1}), \mathbb{Z}) & \longrightarrow & H^j(\mathcal{M}^{k+1}, \mathbb{Z}) & \longrightarrow & \text{Hom}(H_j(\mathcal{M}^{k+1}), \mathbb{Z}) \longrightarrow 0
 \end{array} \quad (3.1)$$

commutes. Since $H^*(\mathcal{M}^k, \mathbb{Z})$ is torsion free, then $\forall n \in \mathbb{N}$, $\exists k_0 = k_0(n)$ such that

$$H_j(\mathcal{M}^k(r, d), \mathbb{Z}) \xrightarrow{\cong} H_j(\mathcal{M}^{k+1}(r, d), \mathbb{Z}) \xrightarrow{\cong} H_j(\mathcal{M}^\infty(r, d), \mathbb{Z})$$

$\forall k \geq k_0$ and $\forall j \leq n \Rightarrow H_j(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0 \ \forall k \geq k_0 \text{ and } \forall j \leq n.$ \square

Proposition 3.12. *For general rank r , denoting $\mathcal{M}^k = \mathcal{M}^k(r, d)$ for simplicity, and $\mathcal{N} = \mathcal{N}(r, d)$ as the moduli of stable bundles, the following diagram commutes*

$$\begin{array}{ccc}
 \pi_1(\mathcal{M}^k) & \xrightarrow{\cong} & \pi_1(\mathcal{M}^{k+1}) \\
 \uparrow \cong & & \uparrow \cong \\
 \pi_1(\mathcal{N}) & \xrightarrow{=} & \pi_1(\mathcal{N})
 \end{array} \quad (3.2)$$

Proof. It is an immediate consequence of the result proved by Bradlow, García-Prada and Gothen [6, Proposition 3.2.] using Morse theory. \square

Proposition 3.13. *For all $k \in \mathbb{N}$, there is an isomorphism between the fundamental group of \mathcal{M}^k and the fundamental group of the direct limit of the spaces $\{\mathcal{M}^k(r, d)\}_{k=0}^\infty$:*

$$\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^\infty).$$

Proof. Using the generalization of Van Kampen's Theorem presented by Fulton [8], and using the fact that $\mathcal{M}^k \hookrightarrow \mathcal{M}^{k+1}$ are embeddings of *Deformation Neighborhood Retracts* (DNR), i.e. every $\mathcal{M}^k(r, d)$ is the image of a map defined on some open neighborhood of itself and homotopic to the identity (see for instance Hausel and Thaddeus [16, (9.1)]), we can conclude that $\pi_1\left(\lim_{k \rightarrow \infty} \mathcal{M}^k\right) = \lim_{k \rightarrow \infty} \pi_1(\mathcal{M}^k).$ \square

We will need the following version of Hurewicz Theorem, presented by Hatcher [13, Theorem 4.37.] (see also James [21]). Hatcher first mentions that, in the relative case when (X, A) is an $(n - 1)$ -connected pair of path-connected spaces, the kernel of the Hurewicz map

$$h : \pi_n(X, A) \rightarrow H_n(X, A; \mathbb{Z})$$

contains the elements of the form $[\gamma][f] - [f]$ for $[\gamma] \in \pi_1(A)$. Hatcher defines $\pi'_n(X, A)$ to be the quotient group of $\pi_n(X, A)$ obtained by factoring out the subgroup generated by the elements of the form $[\gamma][f] - [f]$, or the normal subgroup generated by such elements in the case $n = 2$ when $\pi_2(X, A)$ may not be abelian, then h induces a homomorphism $h' : \pi'_n(X, A) \rightarrow H_n(X, A; \mathbb{Z})$. The general form of Hurewicz Theorem presented by Hatcher deals with this homomorphism:

Theorem 3.14. *If (X, A) is an $(n - 1)$ -connected pair of path-connected spaces, with $n \geq 2$ and $A \neq \emptyset$, then $h' : \pi'_n(X, A) \rightarrow H_n(X, A; \mathbb{Z})$ is an isomorphism and $H_j(X, A; \mathbb{Z}) = 0$ for $j \leq n - 1$.*

Definition 3.15. The *determinant* of a vector bundle $E \rightarrow X$ of rank r is a line bundle giving by the exterior power of the vector bundle. It gives a natural map of the form:

$$\begin{aligned} \det : \mathcal{N} &\longrightarrow \Gamma \\ E &\mapsto \det(E) = \bigwedge^r E \end{aligned}$$

where $\mathcal{N} = \mathcal{N}(r, d)$ is the moduli space of stable bundles $E \rightarrow X$ of rank r and degree d , and $\Gamma = \mathcal{J}^d$ is the Jacobian of X . Fixing a line bundle $\Lambda \rightarrow X$, $\Lambda \in \Gamma$, the fibre $\mathcal{N}_\Lambda = \mathcal{N}_\Lambda(r, d) := \det^{-1}(\Lambda)$ is the moduli space of stable bundles *with fixed determinant*.

There is an important result of Atiyah and Bott [2] that is relevant to mention here:

Theorem 3.16 (Atiyah and Bott [2, (9.12.)]). *The moduli space $\mathcal{N}_\Lambda(r, d)$ of stable bundles of fixed determinant Λ , with $\text{GCD}(r, d) = 1$, is simply connected.*

Remark 3.17. For the case of k -Higgs bundles of fixed determinant $\det(E) = \Lambda$, the moduli space $\mathcal{M}_\Lambda^k(r, d)$ is simply connected by 3.12. So, $\pi_1(\mathcal{M}_\Lambda^k)$ acts trivially on $\pi_n(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k)$.

4 Main Result

Lemma 4.1. *Using the notation $\mathcal{M}^k = \mathcal{M}^k(r, d)$ for the moduli space of stable k -Higgs bundles and for the particular cases of rank $r = 2$ and rank $r = 3$, if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, then for all n exists $k_0 = k_0(n)$ such that $\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0$ for all $k \geq k_0$ and for all $j \leq n$.*

Proof. The proof proceeds by induction on $m \in \mathbb{N}$ for $2 \leq m \leq n$. The first induction step is trivial because

$$\pi_1(\mathcal{N}) = \pi_1(\mathcal{M}) = \pi_1(\mathcal{M}^k) = \pi_1(\mathcal{M}^\infty)$$

by Proposition 3.12. For $m = 2$ we need $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ to be abelian. Consider the sequence

$$\pi_2(\mathcal{M}^\infty) \rightarrow \pi_2(\mathcal{M}^\infty, \mathcal{M}^k) \rightarrow \pi_1(\mathcal{M}^k) \rightarrow \pi_1(\mathcal{M}^\infty) \rightarrow \pi_1(\mathcal{M}^\infty, \mathcal{M}^k) \rightarrow 0$$

where $\pi_2(\mathcal{M}^\infty) \twoheadrightarrow \pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is surjective, $\pi_1(\mathcal{M}^k) \xrightarrow{\cong} \pi_1(\mathcal{M}^\infty)$ are isomorphic, and hence $\pi_1(\mathcal{M}^\infty, \mathcal{M}^k) = 0$. So, $\pi_2(\mathcal{M}^\infty, \mathcal{M}^k)$ is a quotient of the abelian group $\pi_2(\mathcal{M}^\infty)$, and so it is also abelian.

Finally, suppose that the statement is true for all $j \leq m-1$ for $2 \leq m \leq n$. So, $(\mathcal{M}^\infty, \mathcal{M}^k)$ is $(m-1)$ -connected, *i.e.*

$$\pi_j(\mathcal{M}^\infty, \mathcal{M}^k) = 0 \quad \forall j \leq m-1.$$

For $m \geq 2$, by Hurewicz Theorem 3.14,

$$h' : \pi'_m(\mathcal{M}^\infty, \mathcal{M}^k) \xrightarrow{\cong} H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z})$$

is an isomorphism. If $H_n(\mathcal{M}^k(r, d), \mathbb{Z})$ is torsion free $\forall k \in \mathbb{N}$ and $\forall n \in \mathbb{N}$, by Lemma 3.11, $H_m(\mathcal{M}^\infty, \mathcal{M}^k; \mathbb{Z}) = 0$. Hence, if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$ for all $n \in \mathbb{N}$ and for all $k \in \mathbb{N}$, then

$$\pi_m(\mathcal{M}^\infty, \mathcal{M}^k) = \pi'_m(\mathcal{M}^\infty, \mathcal{M}^k) = 0$$

finishing the induction process. \square

Corollary 4.2. *Using the notation $\mathcal{M}^k = \mathcal{M}^k(r, d)$ for the moduli space of stable k -Higgs bundles as above, and for the particular cases of rank $r = 2$ and rank $r = 3$, if $\pi_1(\mathcal{M}^k)$ acts trivially on $\pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$, then for all n exists $k_0 = k_0(n)$ such that*

$$\pi_j(\mathcal{M}^k) \xrightarrow{\cong} \pi_j(\mathcal{M}^\infty)$$

for all $k \geq k_0$ and for all $j \leq n-1$. \square

The main goal here, is to avoid the hypothesis of the trivial action of the fundamental group on the relative homotopy group: $\pi_1(\mathcal{M}^k) \curvearrowright \pi_n(\mathcal{M}^\infty, \mathcal{M}^k)$. So, we want to get the analogue of Lemma 4.1 for \mathcal{M}_Λ^k , the moduli space of k -Higgs bundles with fixed determinant, since \mathcal{M}_Λ^k is simply connected. To do that, we will need the analogue of Corollary 3.2, and then the analogue of Lemma 3.11 also for \mathcal{M}_Λ^k .

The analogue of Corollary 3.2 for \mathcal{M}_Λ^k is not immediate. Note that the group of r -torsion points in the Jacobian:

$$\Gamma = \text{Jac}(r) := \{L \rightarrow X \text{ line bundle} : L^r = \mathcal{O} \text{ is trivial}\}$$

acts on $\mathcal{M}_\Lambda^k(r, d)$ by tensorization:

$$(E, \phi^k) \mapsto (E \otimes L, \phi^k \otimes \text{id}_L).$$

Hence, Γ acts on $H^*(\mathcal{M}_\Lambda^k) \forall k$. This cohomology splits in a Γ -invariant part and in a complement which is called by Hausel and Thaddeus [18] as the “*variant part*”:

$$H^*(\mathcal{M}_\Lambda^k) = H^*(\mathcal{M}_\Lambda^k)^\Gamma \oplus H^*(\mathcal{M}_\Lambda^k)^{\text{var}}. \quad (4.1)$$

This decomposition appears in the various cohomology calculations, see e.g., Hitchin [19] for rank two, Gothen [9] for rank three, Hausel [15] also for rank two, Bento [3] for the explicit calculations for rank two and rank three, and Hausel and Thaddeus [18] for general rank.

The analogue of Corollary 3.2 for \mathcal{M}_Λ^k will be obtained for each of the pieces in the last direct sum (4.1) separately:

- For $H^*(\mathcal{M}_\Lambda^k)^\Gamma$: it follows from the corresponding result for $H^*(\mathcal{M}^k)$ because there is a surjection $H^*(\mathcal{M}^k) \rightarrow H^*(\mathcal{M}_\Lambda^k)^\Gamma$.

For the rank $r = 2$ case, a non-trivial critical submanifold of $\mathcal{M}_\Lambda^k(2, 1)$, is a so-called $(1, 1)$ -VHS:

$$F_{d_1}^k(\Lambda) = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \in \mathcal{M}_\Lambda^k(2, 1) \left| \begin{array}{l} \deg(E_j) = d_j, \quad \text{rk}(E_j) = 1, \\ \varphi_{21}^k : E_1 \rightarrow E_2 \otimes K \otimes L_p^{\otimes k}, \\ E_1 E_2 = \Lambda \end{array} \right. \right\},$$

which is a 2^{2g} -covering with covering group the 2-torsion points in the Jacobian $\Gamma \cong (\mathbb{Z}_2)^{2g}$. Hence, the results of Betti numbers presented by Bento [3, Prop. 2.2.3.] let us conclude the following:

Proposition 4.3. *The cohomology map*

$$H^*(\text{Sym}^m(X)) \rightarrow H^*(F_{d_1}^k(\Lambda))$$

induced by the Γ -covering $F_{d_1}^k(\Lambda) \rightarrow \text{Sym}^m(X)$ where $m = d_2 - d_1 + 2g - 2 + k$, is injective, and its image is the Γ -invariant subgroup $H^(F_{d_1}^k(\Lambda))^\Gamma$. \square*

Corollary 4.4. *There exists a surjection*

$$H^*(\mathcal{M}^k(2, 1)) \twoheadrightarrow H^*(\mathcal{M}_\Lambda^k(2, 1))^\Gamma.$$

\square

When $r = 3$, the group of 3-torsion points in the Jacobian looks like $\Gamma \cong (\mathbb{Z}_3)^{2g}$, and the non-trivial critical submanifolds of $\mathcal{M}_\Lambda^k(3, d)$ are VHS either of type $(1, 2)$, $(2, 1)$ or $(1, 1, 1)$, where the cohomology of the $(1, 2)$ and $(2, 1)$ VHS is invariant under the action of Γ , and the $(1, 1, 1)$ -VHS is a 3^{2g} -covering of $\text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X)$ with covering group $\Gamma \cong (\mathbb{Z}_3)^{2g}$. Hence:

Proposition 4.5.

$$H^*(F_{d_1}^k(\Lambda)) = H^*(F_{d_1}^k(\Lambda))^\Gamma \quad \text{and} \quad H^*(F_{d_2}^k(\Lambda)) = H^*(F_{d_2}^k(\Lambda))^\Gamma$$

where

$$F_{d_1}^k(\Lambda) = \left\{ (E, \Phi^k) = (E_1 \oplus E_2, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \left| \begin{array}{l} \deg(E_1) = d_1, \quad \deg(E_2) = d_2, \\ \text{rk}(E_1) = 1, \quad \text{rk}(E_2) = 2, \\ \varphi_{21}^k : E_1 \rightarrow E_2 \otimes K \otimes L_p^{\otimes k}, \\ E_1 E_2 = \Lambda \end{array} \right. \right\}$$

and

$$F_{d_2}^k(\Lambda) = \left\{ (E, \Phi^k) = (E_2 \oplus E_1, \begin{pmatrix} 0 & 0 \\ \varphi_{21}^k & 0 \end{pmatrix}) \left| \begin{array}{l} \deg(E_2) = d_2, \quad \deg(E_1) = d_1, \\ \text{rk}(E_2) = 2, \quad \text{rk}(E_1) = 1, \\ \varphi_{21}^k : E_2 \rightarrow E_1 \otimes K \otimes L_p^{\otimes k}, \\ E_2 E_1 = \Lambda \end{array} \right. \right\}$$

are the $(1, 2)$ and $(2, 1)$ -VHS of $\mathcal{M}_\Lambda^k(3, d)$ respectively, with

$$\frac{d}{3} \leq d_1 \leq \frac{d}{3} + \frac{2g-2+k}{2} \text{ and } \frac{2d}{3} \leq d_2 \leq \frac{2d}{3} + \frac{2g-2+k}{2}.$$

Furthermore:

$$H^*(F_{m_1 m_2}^k(\Lambda)) = H^*(F_{m_1 m_2}^k(\Lambda))^\Gamma \oplus H^*(F_{m_1 m_2}^k(\Lambda))^{var}$$

and the cohomology map

$$H^*(\text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X)) \rightarrow H^*(F_{m_1 m_2}^k(\Lambda))$$

induced by the Γ -covering $F_{m_1 m_2}^k(\Lambda) \rightarrow \text{Sym}^{m_1}(X) \times \text{Sym}^{m_2}(X)$ where

$$F_{m_1 m_2}^k(\Lambda) = \left\{ (E, \Phi^k) = (E_1 \oplus E_2 \oplus E_3, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_{21}^k & 0 & 0 \\ 0 & \varphi_{32}^k & 0 \end{pmatrix}) \mid \begin{array}{l} \deg(E_j) = d_j, \text{ rk}(E_j) = 1, \\ \varphi_{ij} : E_j \rightarrow E_i \otimes K \otimes L_p^{\otimes k} \\ E_1 E_2 E_3 = \Lambda \end{array} \right\},$$

is the $(1, 1, 1)$ -VHS of $\mathcal{M}_\Lambda^k(3, d)$ with $m_j = d_{j+1} - d_j + 2g - 2 + k$, is injective, and its image is the Γ -invariant subgroup $H^*(F_{m_1 m_2}^k(\Lambda))^\Gamma$. \square

Corollary 4.6. *There exists a surjection*

$$H^*(\mathcal{M}^k(3, d)) \twoheadrightarrow H^*(\mathcal{M}_\Lambda^k(3, d))^\Gamma.$$

\square

The reader may see Bento [3], and also Hausel and Thaddeus [18], for details. Using the results above, we get:

Lemma 4.7. *Let $i_k : \mathcal{M}^k(r, d) \hookrightarrow \mathcal{M}^{k+1}(r, d)$ be the embedding given by the tensorization map $(E, \Phi^k) \mapsto (E, \Phi^k \otimes s_p)$, where s_p is a fixed nonzero section of L_p , with $r = 2$ or $r = 3$. Then, the induced cohomology homomorphism restricted to the Γ -invariant cohomology of the moduli spaces of k -Higgs bundles with fixed determinant Λ*

$$i_k^* : H^*(\mathcal{M}_\Lambda^{k+1}(r, d), \mathbb{Z})^\Gamma \twoheadrightarrow H^*(\mathcal{M}_\Lambda^k(r, d), \mathbb{Z})^\Gamma$$

is surjective.

Proof. It is enough to note that the following diagram

$$\begin{array}{ccc} H^*(\mathcal{M}^{k+1}) & \xrightarrow{i_k^*} & H^*(\mathcal{M}^k) \\ \downarrow & & \downarrow \\ H^*(\mathcal{M}_\Lambda^{k+1})^\Gamma & \xrightarrow{i_k^*} & H^*(\mathcal{M}_\Lambda^k)^\Gamma \end{array} \quad (4.2)$$

commutes, where the top arrow is surjective by Theorem 3.10 and Lemma 3.11, and the descending arrows are surjective because of Corollary 4.4 and Corollary 4.6. \square

- For $H^*(\mathcal{M}_\Lambda)^{var}$: we need to make sure everything works for integer coefficients but it should be ok, since all cohomologies are torsion free.

First, note that with fixed determinant Λ the critical submanifolds of type $(1, 1)$ and $(1, 1, 1)$ are r^{2g} -coverings with covering group $\Gamma \cong (\mathbb{Z}_r)^{2g}$, with $r = 2$ or $r = 3$ (see Bento [3] Prop. 2.2.1. and Lemma 2.4.4.). Furthermore, when $r = 3$ the cohomology of $(1, 2)$ and $(2, 1)$ critical submanifolds is Γ -invariant. Then, only the cohomology of $(1, 1)$ -VHS and $(1, 1, 1)$ -VHS split in the Γ -invariant part and the *variant* complement, for rank $r = 2$ and $r = 3$, respectively. Hence:

$$H^*(\mathcal{M}_\Lambda^k(2, 1))^{var} = \bigoplus_{d_1 > \frac{1}{2}}^{\frac{1+d_k}{2}} H^*(F_{d_1}^k(\Lambda))^{var} \text{ and}$$

$$H^*(\mathcal{M}_\Lambda^k(3, d))^{var} = \bigoplus_{(m_1, m_2) \in \Omega_{d_k}} H^*(F_{m_1 m_2}^k(\Lambda))^{var}$$

where $d_k = \deg(K \otimes L_p^{\otimes k}) = 2g - 2 + k$, $\frac{1}{2} < d_1 < \frac{1+d_k}{2}$ according to Hitchin [19] for $(1, 1)$ -VHS in rank two, and $(m_1, m_2) \in \Omega_{d_k}$ where $M_j := E_j^* E_{j+1} K \otimes L_p^{\otimes k}$, $m_j := \deg(M_j) = d_{j+1} - d_j + d_k$, and the set of indexes

$$\Omega_{d_k} = \left\{ (m_1, m_2) \in \mathbb{N} \times \mathbb{N} \left| \begin{array}{l} 2m_1 + m_2 < 3d_k \\ m_1 + 2m_2 < 3d_k \\ m_1 + 2m_2 \equiv d(3) \end{array} \right. \right\}$$

for $(1, 1, 1)$ -VHS in rank three is described by Bento [3, Prop. 2.3.9.], Gothen [9, Section 3.], Gothen and Zúñiga-Rojas [10, Subsection 5.1], among others.

There are some results appearing in the work of Bento [3] (Lemma 2.2.4. and Prop. 2.2.5 for $\mathcal{M}_\Lambda^k(2, 1)$ and $F_{d_1}^k(\Lambda)$ its $(1, 1)$ -VHS, and Lemma 2.4.4. and Prop. 2.4.5. for $\mathcal{M}_\Lambda^k(3, d)$ and $F_{m_1 m_2}^k(\Lambda)$ its $(1, 1, 1)$ -VHS) where Bento works with Hitchin pairs twisted by a general line bundle L of degree $\deg(L) = d_L$, and the following results below correspond to the particular case of k -Higgs bundles with $L = K \otimes L_p^{\otimes k}$, and hence $d_L = d_k = 2g - 2 + k$:

Lemma 4.8. *Let $F_{d_1}^k$ be a $(1, 1)$ -VHS of $\mathcal{M}^k(2, 1)$ and let $m = d_2 - d_1 + 2g - 2 + k$. Then*

$$H^j(F_{d_1}^k) = 0 \iff j \neq m,$$

and for the case of fixed determinant Λ , the Poincaré polynomial becomes

$$P_t(F_{d_1}^k(\Lambda)) = P_t(\text{Sym}^m(X)) + t^m(2^{2g} - 1) \binom{2g-2}{m}.$$

Proof. See Bento [3, Prop. 2.2.5.]. □

Lemma 4.9. *Let $F_{m_1 m_2}^k(\Lambda)$ be a $(1, 1, 1)$ -VHS of $\mathcal{M}_\Lambda^k(3, d)$. Then*

$$H^i(F_{m_1 m_2}^k(\Lambda)) = 0 \iff i \neq m_1 + m_2,$$

where $m_j = d_{j+1} - d_j + d_k$, and so, the Poincaré polynomial becomes

$$P_t(F_{m_1 m_2}^k(\Lambda)) = P_t(\text{Sym}^{m_1}(X)) P_t(\text{Sym}^{m_2}(X)) + t^{m_1+m_2}(3^{2g} - 1) \binom{2g-2}{m_1} \binom{2g-2}{m_2}.$$

Proof. See Bento [3, Prop. 2.4.5]. \square

Then, in both cases, when $r = 2$ or $r = 3$, the cohomology groups with integer coefficients are torsion free, and it allows us to conclude:

Lemma 4.10. *Let $i_k: \mathcal{M}_\Lambda^k \hookrightarrow \mathcal{M}_\Lambda^{k+1}$ be the embedding given by the tensorization map $(E, \Phi^k) \mapsto (E, \Phi^k \otimes s_p)$, where s_p is a fixed nonzero section of L_p . Then, the induced cohomology homomorphism*

$$i_k^*: H^*(\mathcal{M}_\Lambda^{k+1}, \mathbb{Z})^{var} \rightarrow H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})^{var}$$

is surjective, restricted to the variant complement. \square

This latter method only works with rank $r = 2$ or $r = 3$, but not in general. The difficulty in calculating $H^*(\mathcal{M}_\Lambda)^{var}$ for general rank is explained also on Hausel and Thaddeus [18].

Finally, we may conclude the following:

Corollary 4.11. *Let $i_k: \mathcal{M}_\Lambda^k \hookrightarrow \mathcal{M}_\Lambda^{k+1}$ be the embedding mentioned above. Then, the induced cohomology homomorphism*

$$i_k^*: H^*(\mathcal{M}_\Lambda^{k+1}, \mathbb{Z}) \rightarrow H^*(\mathcal{M}_\Lambda^k, \mathbb{Z})$$

is surjective.

Proof. It is enough to see that the cohomology of \mathcal{M}_Λ^k splits in the Γ -invariant part and the *variant* complement:

$$H^*(\mathcal{M}_\Lambda^k) = H^*(\mathcal{M}_\Lambda^k)^\Gamma \oplus H^*(\mathcal{M}_\Lambda^k)^{var}$$

and so, the result follows from Lemma 4.7 and Lemma 4.10. \square

Lemma 4.12. *For rank $r = 2$ and rank $r = 3$, $\forall n \in \mathbb{N}$, $\exists k_0 = k_0(n)$ such that $H_j(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k; \mathbb{Z}) = 0 \ \forall k \geq k_0$ and $\forall j \leq n$.* \square

Theorem 4.13. *Let $\mathcal{M}_\Lambda^k = \mathcal{M}_\Lambda^k(r, d)$ be the moduli space of stable k -Higgs bundles with fixed determinant $\Lambda \in \Gamma$. Then, for the particular cases of rank $r = 2$ and rank $r = 3$, for all n exists $k_0 = k_0(n)$ such that $\pi_j(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k) = 0$ for all $k \geq k_0$ and for all $j \leq n$.*

Proof. The proof is quite similar to the proof of 4.1, using now Corollary 4.11 and Lemma 4.12, and so, we have a new advantage: \mathcal{M}_Λ^k is simply connected, hence the action $\pi_1(\mathcal{M}_\Lambda^k) \circ \pi_n(\mathcal{M}_\Lambda^\infty, \mathcal{M}_\Lambda^k)$ is trivial. \square

Corollary 4.14. *Let $\mathcal{M}_\Lambda^k = \mathcal{M}_\Lambda^k(r, d)$ be the moduli space of stable k -Higgs bundles with fixed determinant $\Lambda \in \Gamma$ as above. Then, for the cases of rank $r = 2$ and rank $r = 3$, for all n exists $k_0 = k_0(n)$ such that*

$$\pi_j(\mathcal{M}_\Lambda^k) \xrightarrow{\cong} \pi_j(\mathcal{M}_\Lambda^\infty)$$

for all $k \geq k_0$ and for all $j \leq n - 1$. \square

Acknowledgement

Part of this paper is based on my Ph.D. thesis [30] and I would like to thank my supervisor Peter B. Gothen for introducing me to the subject of Higgs bundles, and for all his patience during our illuminating discussions. Mange tak!

I also would like to thank Joseph C. Várilly for his time, listening and reading my results. Thanks for every single advice. Go raibh maith agat!

Last, but not least, I acknowledge the financial support from CIMM, Centro de Investigaciones Matemáticas y Metamatemáticas here in Costa Rica nowadays as a researcher, and also the financial support from FCT, Fundação para a Ciência e a Tecnologia, there in Portugal, when I was a Ph.D. student. ¡Pura Vida! Muito obrigado!

References

- [1] M. F. Atiyah, *K-Theory*, W. A. Benjamin, New York–Amsterdam, 1967.
- [2] M. F. Atiyah and R. Bott, “Yang–Mills equations over Riemann surfaces”, *Phil. Trans. Roy. Soc. London A* **308** (1982), 523–615.
- [3] Bento, S., “Topologia do Espaço Moduli de Fibrados de Higgs Torcidos”, Tese de Doutorado, Universidade do Porto, Porto, Portugal, 2010.
- [4] S. B. Bradlow and O. García-Prada, “Stable triples, equivariant bundles and dimensional reduction”, *Math. Ann.* **304** (1996), 225–252.
- [5] S. B. Bradlow, O. García-Prada and P. B. Gothen, “Moduli spaces of holomorphic triples over compact Riemann surfaces”, *Math. Ann.* **328** (2004), 299–351.
- [6] S. B. Bradlow, O. García-Prada and P. B. Gothen, “Homotopy groups of moduli spaces of representations”, *Topology* **47** (2008), 203–224.
- [7] T. Frankel, “Fixed points and torsion on Kähler manifolds”, *Ann. Math.* **70** (1959), 1–8.
- [8] W. Fulton, *Algebraic Topology, A First Course*, Springer, New York, 1995.
- [9] P. B. Gothen, “The Betti numbers of the moduli space of stable rank 3 Higgs bundles on a Riemann surface”, *Int. J. Math.* **5** (1994), 861–875.
- [10] P. B. Gothen and R. A. Zúñiga-Rojas, “Stratifications on the moduli space of Higgs bundles”, *Portugaliae Mathematica*, to appear.
- [11] P. Griffiths, and J. Harris, *Principles of Algebraic Geometry*, Wiley, New York, 1978.
- [12] G. Harder and M. S. Narasimhan, “On the cohomology groups of moduli spaces of vector bundles on curves”, *Math. Ann.* **212** (1975), 215–248.
- [13] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.

- [14] A. Hatcher, *Vector Bundles and K-Theory*, unpublished.
(<https://www.math.cornell.edu/~hatcher/VBKT/VB.pdf>)
- [15] T. Hausel, “Geometry of Higgs bundles”, Ph.D. thesis, Cambridge, 1998.
- [16] T. Hausel and M. Thaddeus, “Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles”, *Proc. London Math. Soc.* **88** (2004), 632–658.
- [17] T. Hausel and M. Thaddeus, “Relations in the cohomology ring of the moduli space of rank 2 Higgs bundles”, *J. Amer. Math. Soc.* **16** (2003), 303–329.
- [18] T. Hausel and M. Thaddeus, “Mirror symmetry, Langlands duality, and the Hitchin system”, *Invent. Math.* **153** (2003), 197–229.
- [19] N. J. Hitchin, “The self-duality equations on a Riemann surface”, *Proc. London Math. Soc.* **55** (1987), 59–126.
- [20] D. Husemoller, *Fibre Bundles*, third edition, Graduate Texts in Mathematics **20**, Springer, New York, 1994.
- [21] I. M. James, ed., *Handbook of Algebraic Topology*, North-Holland, Amsterdam, 1995.
- [22] I. G. Macdonald, “Symmetric products of an algebraic curve”, *Topology* **1** (1962), 319–343.
- [23] E. Markman, “Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces”, *J. reine angew. Math.* **544** (2002), 61–82.
- [24] E. Markman, “Integral generators for the cohomology ring of moduli spaces of sheaves over Poisson surfaces”, *Adv. Math.* **208** (2007), 622–646.
- [25] V. Muñoz, D. Ortega and M. J. Vázquez-Gallo, “Hodge polynomials of the moduli spaces of pairs”, *Int. J. Math.* **18** (2007), 695–721.
- [26] N. Nitsure, “Moduli space of semistable pairs on a curve”, *Proc. London Math. Soc.* **62** (1991), 275–300.
- [27] S. S. Shatz, “The decomposition and specialization of algebraic families of vector bundles”, *Compos. Math.* **35** (1977), 163–187.
- [28] C. T. Simpson, “Constructing variations of Hodge structures using Yang–Mills theory and applications to uniformization”, *J. Amer. Math. Soc.* **1** (1988), 867–918.
- [29] C. T. Simpson, “Higgs bundles and local systems”, *Publ. Math. IHÉS* ? (1992), 5–95.
- [30] R. A. Zúñiga-Rojas, “Homotopy groups of the moduli space of Higgs bundles”, Ph.D. thesis, Porto, Portugal, 2015.

Ronald A. Zúñiga-Rojas

Centro de Investigaciones Matemáticas

y Metamatemáticas CIMM

Universidad de Costa Rica UCR

San José 11501, Costa Rica

e-mail: ronald.zunigarojas@ucr.ac.cr